

# Beyond the Space-Time Boundary

Michael Heller, Jerzy Król  
Copernicus Center for Interdisciplinary Studies  
Cracow, Poland  
Institute of Physics, University of Silesia

# INTRODUCTION

In relativistic cosmology, the difficulty of providing a mathematical definition of the initial singularity (mathematical counterpart of Big Bang) stems from the fact that the history of the universe is not an invariant concept.

In attempting to solve this problem, we make use of the fact that the histories of massive particles and photons (curves in space-time) are invariant.

# INTRODUCTION

In relativistic cosmology, the difficulty of providing a mathematical definition of the initial singularity (mathematical counterpart of Big Bang) stems from the fact that the history of the universe is not an invariant concept.

In attempting to solve this problem, we make use of the fact that the histories of massive particles and photons (curves in space-time) are invariant.

In GR to say that a space-time is singular means that **there is some positive obstacle that prevents an incomplete curve continuing.**

Roughly speaking there are two kinds of such obstacles:

- 1 some magnitudes, such as curvature or some scalars constructed from it, become unbounded along a timelike curve before it ends, or
- 2 a pathological behaviour of the differential structure of space-time prevents a timelike curve from being prolonged.

We will focus on the latter.

Usually, such a space-time  $M$  is completed to form  $\bar{M} = M \cup \partial M$  where  $\partial M$  is a singular boundary of  $M$ .

The standard geometric tools on  $M$  do not allow “to cross the boundary”. However, the so-called **Synthetic Differential Geometry (SDG)**, a categorical version of standard differential geometry based on **intuitionistic logic**, has at its disposal tools permitting doing so. Owing to the existence of infinitesimals one is able to penetrate “germs of manifolds” that are not visible from the standard perspective.

We present a simple model showing what happens “beyond the boundary” and when the singularity is finally attained. The model is purely mathematical and is mathematically rigorous but, at its present stage, it does not pretend to refer to the physical universe.

Usually, such a space-time  $M$  is completed to form  $\bar{M} = M \cup \partial M$  where  $\partial M$  is a singular boundary of  $M$ .

The standard geometric tools on  $M$  do not allow “to cross the boundary”. However, the so-called **Synthetic Differential Geometry (SDG)**, a categorical version of standard differential geometry based on **intuitionistic logic**, has at its disposal tools permitting doing so. Owing to the existence of infinitesimals one is able to penetrate “germs of manifolds” that are not visible from the standard perspective.

We present a simple model showing what happens “beyond the boundary” and when the singularity is finally attained. The model is purely mathematical and is mathematically rigorous but, at its present stage, it does not pretend to refer to the physical universe.

Usually, such a space-time  $M$  is completed to form  $\bar{M} = M \cup \partial M$  where  $\partial M$  is a singular boundary of  $M$ .

The standard geometric tools on  $M$  do not allow “to cross the boundary”. However, the so-called **Synthetic Differential Geometry (SDG)**, a categorical version of standard differential geometry based on **intuitionistic logic**, has at its disposal tools permitting doing so. Owing to the existence of infinitesimals one is able to penetrate “germs of manifolds” that are not visible from the standard perspective.

We present a simple model showing what happens “beyond the boundary” and when the singularity is finally attained. The model is purely mathematical and is mathematically rigorous but, at its present stage, it does not pretend to refer to the physical universe.

Usually, such a space-time  $M$  is completed to form  $\bar{M} = M \cup \partial M$  where  $\partial M$  is a singular boundary of  $M$ .

The standard geometric tools on  $M$  do not allow “to cross the boundary”. However, the so-called **Synthetic Differential Geometry (SDG)**, a categorical version of standard differential geometry based on **intuitionistic logic**, has at its disposal tools permitting doing so. Owing to the existence of infinitesimals one is able to penetrate “germs of manifolds” that are not visible from the standard perspective.

We present a simple model showing what happens “beyond the boundary” and when the singularity is finally attained. The model is purely mathematical and is mathematically rigorous but, at its present stage, it does not pretend to refer to the physical universe.



# PLAN

- 1 Preliminaries
- 2 Manifold in the Smallest
- 3 A Model
- 4 Comment
- 5 **Appendix**

# Preliminaries

In SDG, one considers various kinds of infinitesimals. Let us denote by  $R$  the real line  $\mathbb{R}$  enriched by infinitesimals. In the present study, we focus on the following ones

$$D = \{x \in R \mid x^2 = 0\},$$

$$D_k = \{x \in R \mid x^{k+1} = 0\}, \quad k = 1, 2, 3, \dots,$$

$$D(n) = \{(x_1, \dots, x_n) \in R^n \mid x_i x_j = 0, \forall i, j = 1, 2, 3, \dots, n\},$$

# Preliminaries

In SDG, one considers various kinds of infinitesimals. Let us denote by  $R$  the real line  $\mathbb{R}$  enriched by infinitesimals. In the present study, we focus on the following ones

$$D = \{x \in R \mid x^2 = 0\},$$

$$D_k = \{x \in R \mid x^{k+1} = 0\}, \quad k = 1, 2, 3, \dots,$$

$$D(n) = \{(x_1, \dots, x_n) \in R^n \mid x_i x_j = 0, \forall i, j = 1, 2, 3, \dots, n\},$$

# Preliminaries

In SDG, one considers various kinds of infinitesimals. Let us denote by  $R$  the real line  $\mathbb{R}$  enriched by infinitesimals. In the present study, we focus on the following ones

$$D = \{x \in R \mid x^2 = 0\},$$

$$D_k = \{x \in R \mid x^{k+1} = 0\}, \quad k = 1, 2, 3, \dots,$$

$$D(n) = \{(x_1, \dots, x_n) \in R^n \mid x_i x_j = 0, \forall i, j = 1, 2, 3, \dots, n\},$$

$D_k(n) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \text{the product of any } k + 1 \text{ of } x_i \text{ is } 0\}$ ,

and finally,

$$(D_\infty)^n = \bigcup_{k=1}^{\infty} D_k(n).$$

Analogously, we can define  $D(V)$  and  $D_k(V)$  for any finite dimensional vector space  $V$ .

$D_k(n) = \{(x_1, \dots, x_n) \in R^n \mid \text{the product of any } k + 1 \text{ of } x_i \text{ is } 0\}$ ,

and finally,

$$(D_\infty)^n = \bigcup_{k=1}^{\infty} D_k(n).$$

Analogously, we can define  $D(V)$  and  $D_k(V)$  for any finite dimensional vector space  $V$ .

$$D_k(n) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \text{the product of any } k + 1 \text{ of } x_i \text{ is } 0\},$$

and finally,

$$(D_\infty)^n = \bigcup_{k=1}^{\infty} D_k(n).$$

Analogously, we can define  $D(V)$  and  $D_k(V)$  for any finite dimensional vector space  $V$ .

We may imagine that infinitesimals constitute the entire world inside every point of  $\mathbb{R}$ , a sort of a fiber over  $x \in \mathbb{R}$ .

Owing to the existence of infinitesimals differentiation becomes a purely algebraic operation and every function is differentiable as many times as required.

This creates a unique opportunity for facing the problem of space-time prolongations and singularities in GR.



In what follows, our important tool is the “ $k$ th order neighbouring relation”, defined as

$$u \sim_k v \Leftrightarrow u - v \in D_k(V).$$

This relation is reflexive and symmetric, but it is not transitive; instead we have

$$(u \sim_k v \wedge v \sim_l w) \Rightarrow (u \sim_{k+l} w).$$

We assume that everything happens in a “suitable” category  $\mathcal{E}$ . “Suitable” means a category equipped, among others with a commutative ring object  $R$ , and usually a topos. We further assume that the category  $\mathbb{M}$  of manifolds (and smooth functions) can be regarded as a subcategory of  $\mathcal{E}$ .

Let  $M$  be an  $n$ -dimensional (formal) manifold regarded as an object of  $\mathcal{E}$ . We are interested in the “smallest neighbourhoods” of  $M$ . A good tool to investigate such neighbourhoods is the neighbourhood relation  $\sim_k$  but it should first be generalised to the manifold context.

Let  $x, y \in M$  and  $k$  a non-negative natural number; the relation  $x \sim_k y$  holds iff there exists a coordinate chart  $f : U \rightarrow M$  such that  $U \subseteq R^n$  is open, and in  $U$  we have  $f(x) \sim_k f(y)$ . If  $k = 1$ , we simply write  $x \sim y$ .

We assume that everything happens in a “suitable” category  $\mathcal{E}$ . “Suitable” means a category equipped, among others with a commutative ring object  $R$ , and usually a topos. We further assume that the category  $\mathbb{M}$  of manifolds (and smooth functions) can be regarded as a subcategory of  $\mathcal{E}$ .

Let  $M$  be an  $n$ -dimensional (formal) manifold regarded as an object of  $\mathcal{E}$ . We are interested in the “smallest neighbourhoods” of  $M$ . A good tool to investigate such neighbourhoods is the neighbourhood relation  $\sim_k$  but it should first be generalised to the manifold context.

Let  $x, y \in M$  and  $k$  a non-negative natural number; the relation  $x \sim_k y$  holds iff there exists a coordinate chart  $f : U \rightarrow M$  such that  $U \subseteq \mathbb{R}^n$  is open, and in  $U$  we have  $f(x) \sim_k f(y)$ . If  $k = 1$ , we simply write  $x \sim y$ .

We assume that everything happens in a “suitable” category  $\mathcal{E}$ . “Suitable” means a category equipped, among others with a commutative ring object  $R$ , and usually a topos. We further assume that the category  $\mathbb{M}$  of manifolds (and smooth functions) can be regarded as a subcategory of  $\mathcal{E}$ .

Let  $M$  be an  $n$ -dimensional (formal) manifold regarded as an object of  $\mathcal{E}$ . We are interested in the “smallest neighbourhoods” of  $M$ . A good tool to investigate such neighbourhoods is the neighbourhood relation  $\sim_k$  but it should first be generalised to the manifold context.

Let  $x, y \in M$  and  $k$  a non-negative natural number; the relation  $x \sim_k y$  holds iff there exists a coordinate chart  $f : U \rightarrow M$  such that  $U \subseteq R^n$  is open, and in  $U$  we have  $f(x) \sim_k f(y)$ . If  $k = 1$ , we simply write  $x \sim y$ .

Let  $M$  be a manifold and  $x, y, z \in M$ . The neighbouring relation  $\sim_k$  satisfies the following conditions

- 1  $x \sim_0 y$  iff  $x = y$  (reflexivity).
- 2  $x \sim_k y$  implies  $y \sim_l x$  if  $k \leq l$  (symmetry),
- 3  $x \sim_k y$  and  $y \sim_l z$  implies  $x \sim_{k+l} z$  (quasi-triangle formula).

Since these conditions are akin to the usual concept of distance, we can define a “quasi-distance” function in the following way

$$\text{dist}(x, y) \leq k \text{ if } x \sim_k y.$$

Since  $D_k(n) \subseteq D_l(n)$  if  $k \leq l$ , the function “dist” determines a “size” of an object  $D_k(n)$ . Let us also notice that this quasi-metric is “quantised” (discrete) since it has its values in  $\mathbb{N}$ .

Let  $M$  be a manifold and  $x, y, z \in M$ . The neighbouring relation  $\sim_k$  satisfies the following conditions

- 1  $x \sim_0 y$  iff  $x = y$  (reflexivity).
- 2  $x \sim_k y$  implies  $y \sim_l x$  if  $k \leq l$  (symmetry),
- 3  $x \sim_k y$  and  $y \sim_l z$  implies  $x \sim_{k+l} z$  (quasi-triangle formula).

Since these conditions are akin to the usual concept of distance, we can define a “quasi-distance” function in the following way

$$\text{dist}(x, y) \leq k \text{ if } x \sim_k y.$$

Since  $D_k(n) \subseteq D_l(n)$  if  $k \leq l$ , the function “dist” determines a “size” of an object  $D_k(n)$ . Let us also notice that this quasi-metric is “quantised” (discrete) since it has its values in  $\mathbb{N}$ .

Let  $M$  be a manifold and  $x, y, z \in M$ . The neighbouring relation  $\sim_k$  satisfies the following conditions

- 1  $x \sim_0 y$  iff  $x = y$  (reflexivity).
- 2  $x \sim_k y$  implies  $y \sim_l x$  if  $k \leq l$  (symmetry),
- 3  $x \sim_k y$  and  $y \sim_l z$  implies  $x \sim_{k+l} z$  (quasi-triangle formula).

Since these conditions are akin to the usual concept of distance, we can define a “quasi-distance” function in the following way

$$\text{dist}(x, y) \leq k \text{ if } x \sim_k y.$$

Since  $D_k(n) \subseteq D_l(n)$  if  $k \leq l$ , the function “dist” determines a “size” of an object  $D_k(n)$ . Let us also notice that this quasi-metric is “quantised” (discrete) since it has its values in  $\mathbb{N}$ .

Now, we define a few key concepts for our further considerations.

## Monad

Let  $x \in M$ . The  $k$ -monad around  $x$  is defined to be

$$\mathcal{M}_k(x) := \{y \in M \mid x \sim_k y\} \subseteq M.$$

If  $k = 1$ , we write  $\mathcal{M}(x)$ .

We also assume that  $\mathcal{M}_\infty$  makes sense.

We obviously have  $y \in \mathcal{M}_k(x) \Leftrightarrow x \in \mathcal{M}_k(y)$ .

If  $f : M \rightarrow N$  is a map between manifolds  $M$  and  $N$  then  $x \sim_k y$  implies  $f(x) \sim_k f(y)$ .



## Df

The “ $k$ th neighbourhood of the diagonal”,  $M_{(k)} \subseteq M \times M$ , is defined to be

$$M_{(k)} := \{(x, y) \in M \times M \mid x \sim_k y\}.$$

If  $V$  is an  $n$ -dimensional vector space, there is a canonical isomorphism

$$M_{(k)} \cong M \times D_k(V)$$

given by

$$(x, y) \mapsto (x, y - x).$$

Consequently, there is an isomorphism between  $\mathcal{M}_k(x)$  and  $D_k(V)$ . Let us also notice that the quasi-metric  $\text{dist}(d_1, d_2) \leq k$  introduces a partial order in  $\mathcal{M}_\infty(x)$  (by inclusions).

## Df

The “ $k$ th neighbourhood of the diagonal”,  $M_{(k)} \subseteq M \times M$ , is defined to be

$$M_{(k)} := \{(x, y) \in M \times M \mid x \sim_k y\}.$$

If  $V$  is an  $n$ -dimensional vector space, there is a canonical isomorphism

$$M_{(k)} \cong M \times D_k(V)$$

given by

$$(x, y) \mapsto (x, y - x).$$

Consequently, there is an isomorphism between  $\mathcal{M}_k(x)$  and  $D_k(V)$ . Let us also notice that the quasi-metric  $\text{dist}(d_1, d_2) \leq k$  introduces a partial order in  $\mathcal{M}_\infty(x)$  (by inclusions).

# Manifold in the Smallest

The existence of infinitesimals essentially enriches the structure of differential manifolds. It enables the following definition

## Formal manifold

An object  $M$  in the category  $\mathcal{E}$  is said to be a  **$k$ -formal  $n$ -dimensional manifold** if, for each  $x \in M$ , there exists a monad  $\mathcal{M}_k(x)$ , isomorphic to  $D_k(n)$ , and a map  $f : \mathcal{M}_k(x) \rightarrow M$ . A bijective map  $D_k(n) \rightarrow \mathcal{M}_k(x)$  onto a monad around  $x$ , mapping  $0$  to  $x$ , is said to be a  $k$ -frame at  $x$ .  $k$  can assume the value  $\infty$ .

If  $\mathcal{M}_\infty(x)$ , we speak of a formal  $n$ -dimensional manifold (without specifying  $k$ ).

It can be easily seen that  $R^n$ , for every  $n$ , is a formal  $n$ -dimensional manifold, and the monad  $\mathcal{M}_\infty(v)$  around  $v \in R^n$  is  $\mathcal{M}(v) = v + D_\infty(n)$ . We have the following natural, but important, results.

- If  $M$  and  $N$  are formal manifolds of dimensions  $m$  and  $n$ , respectively, then  $M \times N$  is an  $(m + n)$ -dimensional formal manifold; and the monad around  $(x, y) \in M \times N$  is  $\mathcal{M}(x) \times \mathcal{M}(y)$  which is isomorphic to  $D_\infty(m + n)$ .
- If  $M$  is a formal  $n$  dimensional manifold, then its tangent bundle  $M^D$  is a  $(m + n)$ -dimensional formal manifold.

It can be easily seen that  $R^n$ , for every  $n$ , is a formal  $n$ -dimensional manifold, and the monad  $\mathcal{M}_\infty(v)$  around  $v \in R^n$  is  $\mathcal{M}(v) = v + D_\infty(n)$ . We have the following natural, but important, results.

- If  $M$  and  $N$  are formal manifolds of dimensions  $m$  and  $n$ , respectively, then  $M \times N$  is an  $(m + n)$ -dimensional formal manifold; and the monad around  $(x, y) \in M \times N$  is  $\mathcal{M}(x) \times \mathcal{M}(y)$  which is isomorphic to  $D_\infty(m + n)$ .
- If  $M$  is a formal  $n$  dimensional manifold, then its tangent bundle  $M^D$  is a  $(m + n)$ -dimensional formal manifold.

# A Model

Let us consider a singular space-time; it is singular in the sense that it contains at least one incomplete curve that cannot be continued in any extension of this space-time. The regular part of this space-time forms a differential manifold  $M$ .

## Completion of space-time

We define the completion of  $M$  as  $\bar{M} = M \cup \partial M$  and call  $\partial M$  the singular boundary of  $M$ . We assume that this boundary is attainable from  $M$ , i.e. that  $M$  is open and dense in  $\bar{M}$ .

Details of this construction are of no importance for our further analysis (there are known several proposals, such as  $g$ -boundary,  $b$ -boundary, causal boundary, and others).

Singular boundary can contain, besides endpoints of inextendible curves that cannot be continued in any extension of space-time, also endpoints of curves that can be continued in some of its extensions, and “points at infinity”.

In what follows, we assume, for simplicity, that  $\partial M$  contains only endpoints of inextendible curves that cannot be continued in any extension of space-time.

For the sake of concreteness let us think about the FLRW space-time with the Big Bang singularity (strong curvature singularity) in the beginning, and let us contemplate the evolution of the universe back in time.

- Everything happens according to the standard cosmological model. The universe shrinks, subsequent cosmic eras succeed each other.
- Finally, the contraction attains the state in which differential properties of space time break down completely.
- The universe leaves the “manifold region”  $M$  and enters its boundary  $\partial M$ . This means that the standard smooth manifold description breaks down, and we assume that at this stage the category  $\mathcal{E}$  takes over.
- The contraction has reached such a degree that infinitesimals enter into play. We thus can employ methods of SDG to gain an insight into what is going on. General picture that emerges is the following.



For the sake of concreteness let us think about the FLRW space-time with the Big Bang singularity (strong curvature singularity) in the beginning, and let us contemplate the evolution of the universe back in time.

- Everything happens according to the standard cosmological model. The universe shrinks, subsequent cosmic eras succeed each other.
- Finally, the contraction attains the state in which differential properties of space time break down completely.
- The universe leaves the “manifold region”  $M$  and enters its boundary  $\partial M$ . This means that the standard smooth manifold description breaks down, and we assume that at this stage the category  $\mathcal{E}$  takes over.
- The contraction has reached such a degree that infinitesimals enter into play. We thus can employ methods of SDG to gain an insight into what is going on. General picture that emerges is the following.

For the sake of concreteness let us think about the FLRW space-time with the Big Bang singularity (strong curvature singularity) in the beginning, and let us contemplate the evolution of the universe back in time.

- Everything happens according to the standard cosmological model. The universe shrinks, subsequent cosmic eras succeed each other.
- Finally, the contraction attains the state in which differential properties of space time break down completely.
- The universe leaves the “manifold region”  $M$  and enters its boundary  $\partial M$ . This means that the standard smooth manifold description breaks down, and we assume that at this stage the category  $\mathcal{E}$  takes over.
- The contraction has reached such a degree that infinitesimals enter into play. We thus can employ methods of SDG to gain an insight into what is going on. General picture that emerges is the following.

For the sake of concreteness let us think about the FLRW space-time with the Big Bang singularity (strong curvature singularity) in the beginning, and let us contemplate the evolution of the universe back in time.

- Everything happens according to the standard cosmological model. The universe shrinks, subsequent cosmic eras succeed each other.
- Finally, the contraction attains the state in which differential properties of space time break down completely.
- The universe leaves the “manifold region”  $M$  and enters its boundary  $\partial M$ . This means that the standard smooth manifold description breaks down, and we assume that at this stage the category  $\mathcal{E}$  takes over.
- The contraction has reached such a degree that infinitesimals enter into play. We thus can employ methods of SDG to gain an insight into what is going on. General picture that emerges is the following.

For the sake of concreteness let us think about the FLRW space-time with the Big Bang singularity (strong curvature singularity) in the beginning, and let us contemplate the evolution of the universe back in time.

- Everything happens according to the standard cosmological model. The universe shrinks, subsequent cosmic eras succeed each other.
- Finally, the contraction attains the state in which differential properties of space time break down completely.
- The universe leaves the “manifold region”  $M$  and enters its boundary  $\partial M$ . This means that the standard smooth manifold description breaks down, and we assume that at this stage the category  $\mathcal{E}$  takes over.
- The contraction has reached such a degree that infinitesimals enter into play. We thus can employ methods of SDG to gain an insight into what is going on. General picture that emerges is the following.

- After crossing  $\partial M$ , domains  $U$  of local charts of the manifold  $M$ ,  $U \rightarrow M$ , become infinitesimal, and the manifold becomes 4-dimensional formal manifold. Local charts are now of the form  $\mathcal{M}_\infty(x) \rightarrow M$ .
- But the universe continues shrinking, and finally its size reduces to a single monad  $\mathcal{M}_\infty(x_0)$ .
- The contraction goes on, but now only in the sense of the metric  $\text{dist}(d_1, d_2) \leq k$ ,  $d_1, d_2 \in \mathcal{M}_\infty(x_0)$ . This means that the differentiability properties are lower and lower, and we obtain a decreasing sequence

$$\mathcal{M}_\infty(x_0), \dots, \mathcal{M}_k(x_0), \mathcal{M}_{k-1}(x_0), \mathcal{M}_{k-2}(x_0), \dots$$

- Finally, when the contraction produces  $\mathcal{M}_0(x_0)$ , the process comes to a halt, since  $\mathcal{M}_0(x_0) = \{y \in R | x_0 \sim_0 y\}$  which, in turn gives  $x_0 = y$ , and all quasi-distances  $\text{dist}(x_0, y)$  reduce to zero.

- After crossing  $\partial M$ , domains  $U$  of local charts of the manifold  $M$ ,  $U \rightarrow M$ , become infinitesimal, and the manifold becomes 4-dimensional formal manifold. Local charts are now of the form  $\mathcal{M}_\infty(x) \rightarrow M$ .
- But the universe continues shrinking, and finally its size reduces to a single monad  $\mathcal{M}_\infty(x_0)$ .
- The contraction goes on, but now only in the sense of the metric  $\text{dist}(d_1, d_2) \leq k$ ,  $d_1, d_2 \in \mathcal{M}_\infty(x_0)$ . This means that the differentiability properties are lower and lower, and we obtain a decreasing sequence

$$\mathcal{M}_\infty(x_0), \dots, \mathcal{M}_k(x_0), \mathcal{M}_{k-1}(x_0), \mathcal{M}_{k-2}(x_0), \dots$$

- .
- Finally, when the contraction produces  $\mathcal{M}_0(x_0)$ , the process comes to a halt, since  $\mathcal{M}_0(x_0) = \{y \in R | x_0 \sim_0 y\}$  which, in turn gives  $x_0 = y$ , and all quasi-distances  $\text{dist}(x_0, y)$  reduce to zero.

- After crossing  $\partial M$ , domains  $U$  of local charts of the manifold  $M$ ,  $U \rightarrow M$ , become infinitesimal, and the manifold becomes 4-dimensional formal manifold. Local charts are now of the form  $\mathcal{M}_\infty(x) \rightarrow M$ .
- But the universe continues shrinking, and finally its size reduces to a single monad  $\mathcal{M}_\infty(x_0)$ .
- The contraction goes on, but now only in the sense of the metric  $\text{dist}(d_1, d_2) \leq k$ ,  $d_1, d_2 \in \mathcal{M}_\infty(x_0)$ . This means that the differentiability properties are lower and lower, and we obtain a decreasing sequence

$$\mathcal{M}_\infty(x_0), \dots, \mathcal{M}_k(x_0), \mathcal{M}_{k-1}(x_0), \mathcal{M}_{k-2}(x_0), \dots$$

- .
- Finally, when the contraction produces  $\mathcal{M}_0(x_0)$ , the process comes to a halt, since  $\mathcal{M}_0(x_0) = \{y \in R | x_0 \sim_0 y\}$  which, in turn gives  $x_0 = y$ , and all quasi-distances  $\text{dist}(x_0, y)$  reduce to zero.

- After crossing  $\partial M$ , domains  $U$  of local charts of the manifold  $M$ ,  $U \rightarrow M$ , become infinitesimal, and the manifold becomes 4-dimensional formal manifold. Local charts are now of the form  $\mathcal{M}_\infty(x) \rightarrow M$ .
- But the universe continues shrinking, and finally its size reduces to a single monad  $\mathcal{M}_\infty(x_0)$ .
- The contraction goes on, but now only in the sense of the metric  $\text{dist}(d_1, d_2) \leq k$ ,  $d_1, d_2 \in \mathcal{M}_\infty(x_0)$ . This means that the differentiability properties are lower and lower, and we obtain a decreasing sequence

$$\mathcal{M}_\infty(x_0), \dots, \mathcal{M}_k(x_0), \mathcal{M}_{k-1}(x_0), \mathcal{M}_{k-2}(x_0), \dots$$

- .
- Finally, when the contraction produces  $\mathcal{M}_0(x_0)$ , the process comes to a halt, since  $\mathcal{M}_0(x_0) = \{y \in R | x_0 \sim_0 y\}$  which, in turn gives  $x_0 = y$ , and all quasi-distances  $\text{dist}(x_0, y)$  reduce to zero.



# Comment

It is instructive to follow the entire process starting from the zero-state. It seems natural to regard increasing sequence of  $k$ s as a sort of quantised time (if this seems to you too farfetched, you can treat the “time” as just a name of a parameter).

However, it is important to notice that  $k$ , in fact, means the “degree of differentiability”, the place at which the Taylor expansion truncates (all higher order terms vanish). Each subsequent instant of this time improves differential properties of the process.

It is astonishing that the transition from  $k = 0$  to  $k = 1$  is so exuberantly reach. Having at our disposal the very first degree of differentiability, we can do large parts of affine geometry, affine connection included, combinatorial differential forms, tangent bundle and lot of differential geometry (in fact, a substantial part of Kock’s seminal SDG monograph is limited to explore the geometry of the first order neighbourhood).

# Comment

It is instructive to follow the entire process starting from the zero-state. It seems natural to regard increasing sequence of  $k$ s as a sort of quantised time (if this seems to you too farfetched, you can treat the “time” as just a name of a parameter).

However, it is important to notice that  $k$ , in fact, means the “degree of differentiability”, the place at which the Taylor expansion truncates (all higher order terms vanish). Each subsequent instant of this time improves differential properties of the process.

It is astonishing that the transition from  $k = 0$  to  $k = 1$  is so exuberantly reach. Having at our disposal the very first degree of differentiability, we can do large parts of affine geometry, affine connection included, combinatorial differential forms, tangent bundle and lot of differential geometry (in fact, a substantial part of Kock’s seminal SDG monograph is limited to explore the geometry of the first order neighbourhood).

# Comment

It is instructive to follow the entire process starting from the zero-state. It seems natural to regard increasing sequence of  $k$ s as a sort of quantised time (if this seems to you too farfetched, you can treat the “time” as just a name of a parameter).

However, it is important to notice that  $k$ , in fact, means the “degree of differentiability”, the place at which the Taylor expansion truncates (all higher order terms vanish). Each subsequent instant of this time improves differential properties of the process.

It is astonishing that the transition from  $k = 0$  to  $k = 1$  is so exuberantly reach. Having at our disposal the very first degree of differentiability, we can do large parts of affine geometry, affine connection included, combinatorial differential forms, tangent bundle and lot of differential geometry (in fact, a substantial part of Kock’s seminal SDG monograph is limited to explore the geometry of the first order neighbourhood).

Of course, when we jump to  $k = 2$ , differentially-geometric properties substantially improve. Some aspects of metric geometry came into force.

For doing  $k$ -jet theory, we evidently need a sufficiently high  $k$ .

Finally, when  $k \rightarrow \infty$ , we end up in  $\mathcal{M}_\infty(x_0)$  and we have the full differentiability. As the universe expands, it goes through the phase of a formal manifold, and when infinitesimals cease to play any role (because of the expansion), the standard smooth manifold regime takes over. In the language of space-time boundary, this means that the universe goes from  $\partial M$  to  $M$ . It is here that we should place the transition from the category  $\mathcal{E}$  to the category SET, and possibly identify this with what physicists call Planck's threshold.

Of course, when we jump to  $k = 2$ , differentially-geometric properties substantially improve. Some aspects of metric geometry came into force.

For doing  $k$ -jet theory, we evidently need a sufficiently high  $k$ .

Finally, when  $k \rightarrow \infty$ , we end up in  $\mathcal{M}_\infty(x_0)$  and we have the full differentiability. As the universe expands, it goes through the phase of a formal manifold, and when infinitesimals cease to play any role (because of the expansion), the standard smooth manifold regime takes over. In the language of space-time boundary, this means that the universe goes from  $\partial M$  to  $M$ . It is here that we should place the transition from the category  $\mathcal{E}$  to the category SET, and possibly identify this with what physicists call Planck's threshold.

Of course, when we jump to  $k = 2$ , differentially-geometric properties substantially improve. Some aspects of metric geometry came into force.

For doing  $k$ -jet theory, we evidently need a sufficiently high  $k$ .

Finally, when  $k \rightarrow \infty$ , we end up in  $\mathcal{M}_\infty(x_0)$  and we have the full differentiability. As the universe expands, it goes through the phase of a formal manifold, and when infinitesimals cease to play any role (because of the expansion), the standard smooth manifold regime takes over. In the language of space-time boundary, this means that the universe goes from  $\partial M$  to  $M$ . It is here that we should place the transition from the category  $\mathcal{E}$  to the category SET, and possibly identify this with what physicists call Planck's threshold.

## Appendix: Through the Boundary

In this appendix, we give a short mathematical description of how the transition through  $\partial M$  could look like. The crucial point is that if we go from space-time  $M$  to its singular boundary  $\partial M$ , we must switch from the category SET of sets and maps between sets as morphisms to a suitable category, to which we have assigned the symbol  $\mathcal{E}$ , the internal logic of which is intuitionistic that enables infinitesimals to appear.

We will describe changing from SET to  $\mathcal{E}$  in a few steps.

## STEP 1:

We move from the description of  $M$  in terms of maps and atlases to the functional description in term of the algebra  $C^\infty(M)$  of smooth functions on  $M$ . This is possible owing to the generalized Gelfand-Naimark theorem:

The category of locally compact Hausdorff spaces and proper continuous maps is anti-equivalent to the category of commutative  $C^*$ -algebras and nondegenerate morphisms.



# Explanations

- A map  $\phi : X \rightarrow Y$  between locally compact Hausdorff spaces is said to be proper if, for any compact  $K \subset Y$ ,  $\phi^{-1}(K)$  is compact in  $X$ .
- A morphism  $\psi : A \rightarrow B$  between  $C^*$ -algebras is said to be nondegenerate, if the linear span of all expressions of the form  $\psi(a)b$ ,  $a \in A$ ,  $b \in B$ , is dense in  $B$ .
- Two categories  $G$  and  $H$  are said to be anti-equivalent (or dual) if there exist contravariant functors  $\alpha : G \rightarrow H$  and  $\beta : G \rightarrow H$  such that  $\alpha \circ \beta$  and  $\beta \circ \alpha$  are naturally isomorphic to  $id_H$  and  $id_G$ , respectively.

**STEP 2:** Let  $C^\infty$  denotes the category, the objects of which are spaces  $\mathbb{R}^n$ ,  $n < 0$ , and the arrows are smooth functions.

Df

The category of smooth algebras has as its objects finite product preserving functors  $A : C^\infty \rightarrow \text{SET}$ , and as its arrows natural transformations between them.

If  $M \in \mathbb{M}$  is a smooth manifold then the functor

$$C^\infty(M) = \text{Hom}_{\mathbb{M}}(M, -)$$

is a smooth algebra  $C^\infty(M)$ .

Considering an object (here a smooth manifold) in another category may not be a superficial change. Changing a categorical context can not only provide new tools of investigation, but can also affect properties of the object itself. Let us pursue this line of research.

**STEP 2:** Let  $C^\infty$  denotes the category, the objects of which are spaces  $\mathbb{R}^n$ ,  $n < 0$ , and the arrows are smooth functions.

Df

The category of smooth algebras has as its objects finite product preserving functors  $A : C^\infty \rightarrow \text{SET}$ , and as its arrows natural transformations between them.

If  $M \in \mathbb{M}$  is a smooth manifold then the functor

$$C^\infty(M) = \text{Hom}_{\mathbb{M}}(M, -)$$

is a smooth algebra  $C^\infty(M)$ .

Considering an object (here a smooth manifold) in another category may not be a superficial change. Changing a categorical context can not only provide new tools of investigation, but can also affect properties of the object itself. Let us pursue this line of research.

**STEP 2:** Let  $C^\infty$  denotes the category, the objects of which are spaces  $\mathbb{R}^n$ ,  $n < 0$ , and the arrows are smooth functions.

Df

The category of smooth algebras has as its objects finite product preserving functors  $A : C^\infty \rightarrow \text{SET}$ , and as its arrows natural transformations between them.

If  $M \in \mathbb{M}$  is a smooth manifold then the functor

$$C^\infty(M) = \text{Hom}_{\mathbb{M}}(M, -)$$

is a smooth algebra  $C^\infty(M)$ .

Considering an object (here a smooth manifold) in another category may not be a superficial change. Changing a categorical context can not only provide new tools of investigation, but can also affect properties of the object itself. Let us pursue this line of research.

For every smooth manifold  $M$ , the smooth algebra  $C^\infty(M)$  is finitely presented.

A smooth algebra  $A$  is said to be finitely generated if it is of the form  $C^\infty(\mathbb{R}^n)/I$ , for  $n \in \mathbb{N}$  and an ideal  $I$ ; if additionally  $I$  is finitely generated,  $A$  is said to be finitely presented.

Df

Finitely generated smooth algebras and  $C^\infty$ -homomorphisms between them as morphisms form a category denoted by  $\text{FGAlg}$ .

### STEP 3:

#### Df

We define the category LOC (of loci) as the opposite category with respect to FGAlg. The objects of LOC are the same as those of FGAlg (if  $A \in \text{FGAlg}$ , we shall write  $!A$  when  $A$  is considered as an object of LOC), but morphisms are reversed.

Since the functor from the manifold category to  $C^\infty\text{-Alg}$ , given by  $M \mapsto C^\infty(M)$ , is contravariant, to obtain the correct variance, we switch to the category LOC.

And indeed, we have the covariant functor  $s : \mathbb{M} \rightarrow \text{LOC}$ , given by  $M \mapsto !C^\infty(M)$  which, importantly, is full and faithful.

### STEP 3:

#### Df

We define the category LOC (of loci) as the opposite category with respect to FGAlg. The objects of LOC are the same as those of FGAlg (if  $A \in \text{FGAlg}$ , we shall write  $!A$  when  $A$  is considered as an object of LOC), but morphisms are reversed.

Since the functor from the manifold category to  $C^\infty\text{-Alg}$ , given by  $M \mapsto C^\infty(M)$ , is contravariant, to obtain the correct variance, we switch to the category LOC.

And indeed, we have the covariant functor  $s : \mathbb{M} \rightarrow \text{LOC}$ , given by  $M \mapsto !C^\infty(M)$  which, importantly, is full and faithful.

This switching the categorical environment produces radical changes.

- A new element that appears in this context is the so-called Weil algebra. It is a finite dimensional  $\mathbb{R}$ -algebra  $W$  having a maximal ideal  $I$  such that  $W/I \simeq \mathbb{R}$  with  $I^n = 0$  for some  $n \in \mathbb{N}$ . It has a unique smooth algebra structure and is finitely presented.
- Objects of LOC (smooth loci) corresponding to Weil algebras are infinitesimal spaces.
- In this way, infinitesimals appear in our model. We immediately have:  $C^\infty(\mathbb{R}) \in \text{LOC}$  is a real line enriched with infinitesimals which we have denoted by  $R$ .



We now are “well inside” the boundary  $\partial M$ . As the universe continues shrinking, domains of local maps become infinitesimals, and the usual manifold is replaced by a formal manifold. Finally, when shrinking still progresses, everything is reduced to a single monad  $\mathcal{M}_\infty(x_0)$ .

Usually, to improve geometric properties of a given model, one once again changes the category  $\text{LOC}$  to the category  $\text{SET}^{\text{LOC}^{\text{op}}}$  of presheaves on  $\text{LOC}$  or to some of its subcategories. In our case, this does not seem indispensable since  $\text{LOC}$  has good properties if limited to “sufficiently small” spaces.

We now are “well inside” the boundary  $\partial M$ . As the universe continues shrinking, domains of local maps become infinitesimals, and the usual manifold is replaced by a formal manifold. Finally, when shrinking still progresses, everything is reduced to a single monad  $\mathcal{M}_\infty(x_0)$ .

Usually, to improve geometric properties of a given model, one once again changes the category LOC to the category  $\text{SET}^{\text{LOC}^{op}}$  of presheaves on LOC or to some of its subcategories. In our case, this does not seem indispensable since LOC has good properties if limited to “sufficiently small” spaces.